Universidad de Costa Rica
Escuela de Matemática
Proyecto MATEM

Proyecto MATEM: Cálculo I
IV Examen Parcial - Solucionario

Total de puntos: 50

Parte única: Desarrollo

1) (34 puntos) Calcule las siguientes integrales:

a) (8 puntos) $\int \ln^2(x)dx$

Sean:

$u=\ln^2(x) \quad dv=dx$
$du=\frac{2\ln(x)}{x} \quad v=x$ Entonces:

$$\int \ln^2(x)dx = x\ln^2(x) - \int \frac{2\cancel{x}\ln(x)}{\cancel{x}}dx$$
$$= x\ln^2(x) - 2\int \ln(x)dx$$

Sean:

$u=\ln(x) \quad dv=dx$
$du=\frac{1}{x}dx \quad v=x$ Entonces:

$$= x\ln^2(x) - 2\left[x\ln(x) - \int \cancel{x}\frac{1}{\cancel{x}}dx\right]$$
$$= x\ln^2(x) - 2\left[x\ln(x) - \int dx\right]$$
$$= x\ln^2(x) - 2\left[x\ln(x) - x\right] + C$$
$$= x\ln^2(x) - 2x\ln(x) - 2x + C$$

b) (8 puntos) $\int_0^1 3^{\sqrt{2x+1}}dx$

Sean:

$u=\sqrt{2x+1} \quad du=\frac{dx}{\sqrt{2x+1}}$
$u^2=2x+1 \quad \sqrt{2x+1}du=dx$
$|u|du=dx$

$x=0 \to u=0$
$x=1 \to u=\sqrt{3}$

$$\int_0^1 3^{\sqrt{2x+1}}dx = \int_0^{\sqrt{3}} 3^u u\,du$$

Sean:

$w = u \qquad dv = 3^u$

$dw = du \qquad v = \frac{3^u}{\ln(3)}$

Entonces:

$$\int_0^{\sqrt{3}} 3^u u\,du = u\frac{3^u}{\ln 3} - \int_0^{\sqrt{3}} \frac{3^u}{\ln(3)}du$$

$$= \left[u\frac{3^u}{\ln 3} - \frac{3^u}{\big(\ln(3)\big)^2}\right]\Bigg|_1^{\sqrt{3}}$$

$$= \sqrt{3}\frac{3^{\sqrt{3}}}{ln(3)} - \frac{3^{\sqrt{3}}}{\big(\ln(3)\big)^2} - \frac{3}{\ln(3)} + \frac{3}{\big(\ln(3)\big)^2}$$

$$= \frac{\sqrt{3}\cdot 3^{\sqrt{3}}\ln(3) - 3^{\sqrt{3}}}{\big(\ln(3)\big)^2} - \frac{3\ln(3) - 3}{\big(\ln(3)\big)^2}$$

$$= \frac{3^{\sqrt{3}}\left(\sqrt{3}\cdot\ln(3) - 1\right)}{\big(\ln(3)\big)^2} - \frac{3\left(\ln(3) - 1\right)}{\big(\ln(3)\big)^2}$$

c) **(9 puntos)** $\displaystyle\int \frac{\sqrt{9x^2-4}}{x}dx$

Sea: $x = \frac{2}{3}\sec\theta \quad dx = \frac{2}{3}\sec\theta\tan\theta$

Entonces:

$$\int \frac{\sqrt{9x^2-4}}{x}dx = \int \frac{\sqrt{4\sec^2\theta - 4}}{\cancel{\frac{2}{3}\sec\theta}} \cdot \cancel{\frac{2}{3}\sec\theta}\tan\theta\,d\theta$$

$$= \int \sqrt{4(\sec^2\theta - 1)}\cdot\tan\theta\,d\theta$$

$$= \int 2\sqrt{\tan^2\theta}\cdot\tan\theta\,d\theta$$

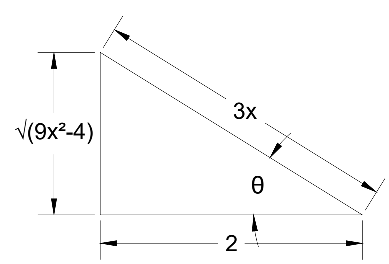
$$= 2\int \tan^2\theta\,d\theta$$

$$= 2\int (\sec^2\theta - 1)\,d\theta$$

$$= 2\left[\int \sec^2\theta\,d\theta - \int d\theta\right]$$

$$= 2\tan\theta - 2\theta + C$$

$$= \frac{\cancel{2}\sqrt{9x^2-4}}{\cancel{2}} - \text{arcsec}\left(\frac{3}{2}x\right) + C$$



$\tan\theta = \frac{\sqrt{9x^2-4}}{2}$

$\sec\theta = \frac{3x}{2} \Rightarrow \theta = \text{arcsec}\left(\frac{3}{2}x\right)$

d) (9 puntos) $\int \frac{2x^2-1}{x\left(x^2+2x+2\right)}dx$

Fracciones parciales:

$$\frac{2x^2-1}{x\left(x^2+2x+2\right)} = \frac{A}{x} + \frac{Bx+C}{\left(x^2+2x+2\right)}$$
$$\frac{2x^2-1}{\cancel{x\left(x^2+2x+2\right)}} = \frac{Ax^2+2Ax+2A+Bx^2+Cx}{\cancel{x\left(x^2+2x+2\right)}}$$

$2x^2-1 = (A+B)x^2 + (2A+C)x + 2A$

$A = \frac{-1}{2}$
$A+B = 2 \Rightarrow B = 2 + \frac{1}{2} = \frac{5}{2}$
$2A + C = 0 \Rightarrow C = -2\left(\frac{-1}{2}\right) = 1$

Por lo tanto:

$$\int \frac{2x^2-1}{x\left(x^2+2x+2\right)}dx = \int \frac{\frac{-1}{2}}{x}dx + \int \frac{\frac{5}{2}x+1}{x^2+2x+2}$$
$$= \frac{-1}{2}\ln|x| + \frac{5}{4}\int \frac{2x+\frac{4}{5}}{\left(x^2+2x+2\right)}$$
$$= \frac{-1}{2}\ln|x| + \frac{5}{4}\int \frac{2x+2}{\left(x^2+2x+2\right)}dx - \frac{6}{5}\int \frac{1}{\left(x^2+2x+2\right)}dx$$
$$= \frac{-1}{2}\ln|x| + \frac{5}{4}\ln\left|x^2+2x+2\right| - \frac{6}{5}\arctan(x+1)$$

2) (10 puntos) Determine si la siguiente integral converge o diverge. En caso de ser convergente, calcule su valor.

$$\int_0^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx$$

$$\int_0^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx = \int_0^a \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx + \int_a^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx$$

$\forall a \in \mathbb{R}, \text{a} > 0$

En particular, con a=1:

$$\int_0^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx = \int_0^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx + \int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx$$

Sean:

$u = -\sqrt{x} \quad du = \frac{-1}{2\sqrt{x}}dx$
$-2du = \frac{dx}{\sqrt{x}}$

Entonces:

$$\int \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx = \int -2e^u du$$
$$= -2e^u + C$$

Luego:

$$\int_0^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx = \lim_{t\to 0^+}\int_t^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}}$$

$$\lim_{t\to 0^+}\left[-2e^{-\sqrt{x}}\right]\Big|_t^1$$

$$\lim_{t\to 0^+}\left[-2e^{-\sqrt{1}}+2e^{-\sqrt{t}}\right] = -\frac{2}{e}+2$$

$$\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx = \lim_{t\to \infty}\int_1^t \frac{e^{-\sqrt{x}}}{\sqrt{x}}$$

$$\lim_{t\to \infty}\left[-2e^{-\sqrt{x}}\right]\Big|_1^t$$

$$\lim_{t\to 0^+}\left[-2e^{-\sqrt{t}}+2e^{-\sqrt{1}}\right] = -\frac{2}{e}$$

Por lo tanto:

$$\int_0^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx = \cancel{-2e^{-1}}+2+\cancel{2e^{-1}} = 2$$

3) (6 puntos) Demuestre, utilizando fórmulas de ángulos dobles, que:

$$\int \left(\operatorname{sen}^2(x)\cdot\cos^2(x)\right)dx = \frac{1}{8}x - \frac{\operatorname{sen}(4x)}{32}+C$$

$$\int \left(\operatorname{sen}^2(x)\cdot\cos^2(x)\right)dx$$

$$= \int \left(\frac{1-\cos(2x)}{2}\cdot\frac{1+\cos(2x)}{2}\right)dx$$

$$= \int \frac{1-\cos^2(2x)}{4}$$

$$= \int \frac{1}{4}dx - \int \frac{\cos^2(2x)}{4}dx$$

$$= \frac{1}{4}x - \frac{1}{4}\int \cos^2(2x)dx$$

Dado que:

$\cos^2(2x) = \frac{1+\cos(4x)}{2}$

Entonces:

$$= \frac{1}{4}x - \frac{1}{4}\int \frac{1+\cos(4x)}{2}dx$$

$$\frac{1}{4}x - \frac{1}{8}\int \left[1+\cos(4x)\right]dx$$

$$= \frac{1}{4}x - \frac{1}{8}\left(x+\frac{\operatorname{sen}(4x)}{4}\right)+C$$

$$= \frac{1}{8}x - \frac{1}{32}\operatorname{sen}(4x)+C$$